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Abstract

A fixed-point free involution on a finite set S is defined as a bijection $I : S \to S$ such as $\forall e \in S, I(I(e)) = e$ and $\forall e \in S, I(e) \neq e$.

In this article, the fixed-point free involutions are represented as partitions of the set S, and some properties linked to this representation are exhibited.

Then an optimal algorithm to list all the fixed-point free involutions is presented. Its soundess relies on the representation of the fixed-point free involutions as partitions.

Finally, an implementation of the algorithm is proposed, with an effective data representation.

Keywords

Algorithm, Fixed-point free involutions, Partitions, Recursion

1 Introduction

A fixed-point free involution involution on a finite set is a function which can be defined as follows :

$$\forall e \in S, I(I(e)) = e$$

$$\forall e \in S, I(e) \neq e$$

Some recent cryptanalysis methods are based on fixed-point free involutions on finite sets. Indeed, such functions can be seen as mixing functions with a structural weakness, which make them trivially invertible.

Such functions can be used instead of a cryptographically robust functions, in ordrer to study the behaviour of a cryptographyc algorithm [Poinsot (2006)]. An other possible cryptographic attack is to find weak keys, such as the algorithm is equivalent to a fixed-point free involution [Prissette (2004)].

Obviously, an algorithm can be use to list all the permutations π of the finite set S [Dijkstra (1997)]. For each permutation, I as $\forall e \in S, I(e) = \pi_e$, it is easy to check the constraint of involution ($\forall e \in S, I(I(e)) = e$) and the absence of fixed-point.

However, cryptography use large finite sets and using an algorithm to list all the permutation is a waste of time, because the number of fixed-point free involutions on a set roughly equals the square root of the number of permutations on the set.

In the first part of this article, some properties of the fixed-point free involutions are presented and a quick proof is given for each of them. Then, in a second part, an algorithm to list all fixed-point free involutions on a finite set is described. Finally, an effective data representation is proposed in an example of implementation of the algorithm.

2 Properties of the Fixed-Point Free Involutions

2.1 Fixed-Point Free Involutions and Partitions

Let I be a fixed-point free involution on S and define P_I as follows :

$$P_I = \{\{e, I(e)\}, \forall e \in S\}$$

Obviously, as I is a fixed-point free involution , e and $I(e) = e'$ generate the same subset $\{e, I(e)\} = \{I(I(e)), e')\} = \{I(e'), e'\} = \{e', I(e')\}$. Every element of S is a element of a single element of P_I. Thus, the fixed-point free involution I defines the set P_I as a partition of S, such as the cardinality of every subset of P_I is 2.

Conversely, given a partition P_I of S, such as the cardinality of every element of P_I is 2, the involution I can be defined as follows :

$$\forall \{e, e'\} \in P_I, \begin{cases} I(e) = e' \\ I(e') = e \end{cases}$$

As every element of S is an element of a single element of P_I, then $I(I(e)) = I(e') = e$. Moreover, as the cardinality of every element of P_I is 2, thus $\forall_{e \in S}, I(e) \neq e$. So I is the fixed-point free involution defined by P_I.

The fixed-point free involution I can be represented in a single way as the partition P_I. This property will be use to represent fixed-point free involution in a convenient way.

2.2 Fixed-Point Free Involutions and Union

Let I be a fixed-point free involution on S. Let i and j be such as $i \notin S$ and $j \notin S$ and $i \neq j$. Considering the partition P_I of S, associated to I, $P_{I'} = P_I \cup \{\{i, j\}\}$ is a partition of $S \cup \{i, j\}$ and the cardinality of every subsets is 2. So $P_{I'}$ can be used to represent a fixed-point free involution I' on $S \cup \{i, j\}$, defined as follows :

$$\begin{cases} \forall e \in S, I'(e) = I(e) \\ I'(i) = j \\ I'(j) = i \end{cases}$$

The main idea of the algorithm is to build a fixed-point free involution, with a fixed-point free involution on a smaller set.

2.3 Fixed-Point Free Involutions and Bijections

Let I be a fixed-point free involution on S,
Let B be a bijection from S to S'
Let's define I' from S' to S' as follows :

$$I'(e') = B \circ I \circ B^{-1}(e')$$

Let's prove that I' is a fixed-point free involution. First, Let's prove that I' is an involution.

$$\begin{aligned} & I'(I'(e')) = I' \circ B \circ I \circ B^{-1}(e') \\ \Leftrightarrow\ & I'(I'(e')) = B \circ I \circ B^{-1} \circ B \circ I \circ B^{-1}(e') \\ \Leftrightarrow\ & I'(I'(e')) = B \circ I \circ I \circ B^{-1}(e') \\ \Leftrightarrow\ & I'(I'(e')) = B \circ B^{-1}(e') \\ \Leftrightarrow\ & I'(I'(e')) = e' \end{aligned}$$

So I' is an involution.

Now, let's prove that I' is fixed-point free. If $I'(e') = e'$, then

$$\begin{aligned} & B^{-1} \circ B \circ I \circ B^{-1}(e') = B^{-1}(e') \\ \Leftrightarrow\ & I \circ B^{-1}(e') = B^{-1}(e') \\ \Leftrightarrow\ & I(e) = e \text{ with } e = B^{-1}(e') \end{aligned}$$

However, I is fixed-point free. So the previous equality is false, and I' is a fixed-point free involution.

This property is useful to build fixed-point free involution on any set with an even cardinality : one can build a fixed-point free involution on a simple set with the same cardinality, then use a bijection to map this fixed-point free involution onto the wanted set.

Without loss of generality, S is defined as $\{1, 2, 3, ..2n\}$ until the end of this article.

3 Algorithm

3.1 Bijections Family

For the purpose of the algorithm, a family of bijections from $S = \{1, 2, .., 2k\}$ to $\{2, .., 2k + 2\} \setminus i$ is needed, with i in $\{2, 2, .., 2k + 2\}$.

Although many families of bijections may be used, the following one is chosen :

$$B_i(x) = \begin{cases} x + 2 \text{ if } x + 1 \geq i \\ x + 1 \text{ if } x + 1 < i \end{cases}$$

Every element of the family is an easy-to-compute, easy-to-invert, increasing function. None of these properties is mandatory; however, they are useful for saving time and space.

Obviously, the inverse function of B_i is :

$$B_i^{-1}(x) = \begin{cases} x - 1 \text{ if } x + 1 \leq i \\ x - 2 \text{ if } x + 1 > i \end{cases}$$

3.2 Presentation of the Algorithm

The goal of the algorithm is to construct the set of the fixed-point free involutions on the finite set $S = \{1, 2, .., n\}$, with even cardinality. The main idea is to start with the simple fixed-point free involution represented by the partition $\{\{\}\}$ then, the size of the set is increased using the "Union Property" and the "Bijection Property".

This is a recursive process : given a fixed-point free involution on $\{1, .., k\}$, a bijection is used to get an fixed-point free involution on $\{2, .., 2k+2\} \setminus i$, then the union property is used to add the set $\{1, i\}$ and get a fixed-point free involution on $(\{2, .., 2k + 2\} \setminus i) \cup \{1, i\} = \{1, .., k\}$.

Here is a description of the algorithm, as a recursive function.

function fpfi(n,S)

// n : cardinal of the final set

// S : current set (initial value = ∅)

if $(|S| = n)$ **then**

output S

else

for i=2 **to** n

$S' = \{\{1, i\}\}$

forall $\{e, e'\} \in S$

$S' = S' \cup \{\{B_i(e), B_i(e')\}\}$

end forall

fpfi(n,S')

end for

end if

end

3.3 Example

Here is a quick description of the building of one of the involutions on the set $\{1,..,6\}$, knowing an involution on the set $\{1,..,4\}$, for example the involution I such as :

$$P_I = \{\{1,3\},\{2,4\}\}$$

There are 5 involutions built on I, each of them includes $\{1,i\}$ with a different value of i in $\{2,..,6\}$. For each of these involutions, the associated partition is built using I and B_i.

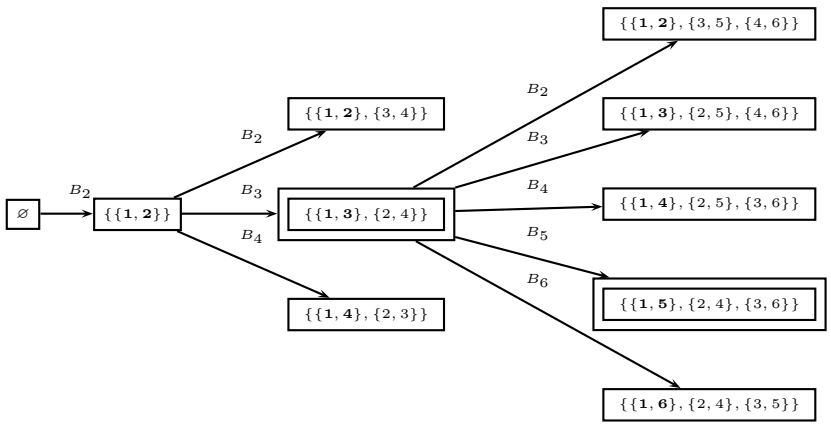
For example, for $i = 5$, the elements of the partition are :

- the element $\{1,5\}$.
- the elements built with B_5 and P_I :
  - from $\{1,3\}$, compute $\{B_5(1), B_5(3)\} = \{2,4\}$
  - from $\{2,4\}$, compute $\{B_5(2), B_5(4)\} = \{3,6\}$

The resulting involution is associated with the set of these three sets :

$$\{\{1,5\},\{2,4\},\{3,6\}\}$$

The following tree shows how some of the fixed-point free involution on $\{1,..,6\}$ are built.



4 Implementation

4.1 Data Representation

As previously shown, a fixed-point free involution I on the set S can be represented as a partition P_I of S such as the cardinality of every element of P_I is 2.

Let's $\mu(P_I)$ be the set of elements of S defined as follows :

$$\mu(P_I) = \{min(i,j), \forall_{\{i,j\} \in P_I}\}$$

The algorithm represents the partition P_I as an 2n-element array T.

$$\begin{cases} \forall_{0 \leq k < n}, & T[2k] \in \mu(P_I) \\ \forall_{0 \leq k < n-1}, & T[2k] > T[2k+2] \\ \forall_{0 \leq k < n-1}, & T[2k+1] = I(T[2k]) \end{cases}$$

Simply speaking, the odd-indexed elements of T are, decreasingly sorted, the set of the lowest elements of each set of the partition P_I. The even-indexed elements of T are the values associated by I to the odd-indexed elements.

4.1.1 Example

Let I be the fixed-point free involution on $\{1..6\}$ such as $I(x) = 7 - x$. This fixed-point free involution is represented by the partition P_I :

$$P_I = \{\{2,5\},\{3,4\},\{6,1\}\}$$

With this partition, $\mu(P_I)$ is defined as :

$$\mu(P_I) = \{2,3,1\}$$

This partition P_I (and so the fixed-point free involution I) is represented as the array T :

$$T = [3,4,2,5,1,6]$$

4.2 Union operator

The purpose of the proposed data representation is to speed up the calculation of the Union operator. Actually, with this representation, the bijection do not destroy the order of the element, and the new couple $(1, I(1))$ is simply merged at the end of the array.

In many practical implementations, it can be effectively done by allocating an array as large as the size of the set, and recursively filling the array from left to right.

An example of such an implementation is given in Appendix A.

5 Conclusion

The algorithm presented in this article was designed to fit cryptographic needs of an effective algorithm to list all the fixed-point free involutions on a finite set. However, the use of such functions is not restricted to cryptographic researches, and the algorithm is generic.

References

[Levitin (2002)] Levitin, A.V. (2002). Introduction to the Design & Analysis of Algorithms. Addison Wesley. ISBN 0201743957.

[Conner and Floyd (1960)] Conner, P.E. and Floyd, E. E. (1960) Fixed point free involutions and equivariant maps. In *Journal: Bull. Amer. Math. Soc.* Volume 66 Pages 416-441

[Poinsot (2006)] Poinsot, L. (2006). Boolean Bent Functions in Impossible Cases: Odd and Plane Dimensions. In *IJCSNS International Journal of Computer Science and Network Security.* Volume 6, No.8A

[Prissette (2004)] Prissette, C. Weak keys of graph cryptography. In *Information and Communication Technologies: From Theory to Applications, 2004. Proceedings. 2004 International Conference.* Pages 419-420

[Dijkstra (1997)] E. W. Dijkstra A Discipline of Programming. Prentice-Hall, 1997.